The Instantaneous Capital Market Line[1]

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Abstract

We show that if the intercept and slope of the instantaneous capital market line are deterministic, then investors will not hold any hedge portfolios in the sense of Merton [9, 1973], [11, 1990]. They will choose portfolios that plot on the capital market line, and they will slide up and down the capital market line over time as their wealth and risk tolerance change. This result allows us to aggregate over investors and derive a single factor CAPM where the first and second moments of security returns may change stochastically over time and markets are potentially incomplete.

1 Introduction

This paper shows that if the intercept and slope of what we call the instantaneous capital market line (ICML) are deterministic, then investors will not hedge against changes in the means, variances or covariances of security returns.

We formulate our result in the form of a two-fund separation theorem. It says that if the ICML is deterministic, then investors will simply hold a possibly time-varying combination of two funds that span the ICML: the riskless asset and the logarithmic portfolio. Investors will place themselves along the ICML, and they will slide up and down the ICML over time as their wealth and risk tolerance change.

The theorem holds in a general framework with securities whose rates of return have stochastically time-varying first and second moments which do not have to be functions of a Markovian vector of state variables. The investor's utility is a state-independent function of his consumption over time and of his final wealth at the horizon time. Markets are not assumed to be dynamically complete.

The key to the proof is to create a simplified securities market model with only two assets, the money market account and the logarithmic portfolio, and then show that risk averse investors will prefer payoffs that can be replicated within this model.

The capital market line (CML) has always played a central role in static mean-variance portfolio theory. It would be graphed and used to visualize the determination of the optimal portfolio. Merton [8, 1971] showed that if the interest rate and all the first and second moments are constant, then the ICML has the same role in continuous time as the CML has in a static model. Our result implies that the ICML plays this same role in a wider set of circumstances.

The results in the literature that are closest to ours are found in Karatzas, Lehoczky, Shreve, and Xu [5, 1991] and Ocone and Karatzas [14, 1991]. Both assume complete markets, and Karatzas et al. [5, 1991] also assume a power utility function. The mathematical methods they use are quite different from ours: Ocone and Karatzas use Malliavin calculus, whereas Karatzas et al. do an explicit calculation relying on the parameter of the utility function. Both find that if the interest rate and the entire vector of prices of risk (as opposed to the length of this vector, which is the slope of the ICML) are deterministic, then investors do not need to hedge. This result cannot be recast in terms of the slope of the ICML and thus cannot be used to establish the ICML as the investment opportunity set.

Our result is similar in spirit to those of Constantinides [3, 1980]. He identifies some circumstances in which investors will not hold hedge portfolios even though some asset returns may be non-stationary, and in particular, even though some assets may have stochastically time-varying means, variances, and correlations. Specifically, he shows that this is true in equilibrium if the investors' utility functions have the aggregation property and all assets in positive supply have stationary returns. Our results do not assume equilibrium or aggregating utility functions, and our assumptions about returns concern only the dynamics of the ICML.

The two fund separation theorem allows us subsequently to aggregate over investors and derive a single factor CAPM where the first and second moments of security returns may change stochastically over time and markets are potentially incomplete.

In equilibrium, under the assumptions of our two-fund separation theorem, the market portfolio will be proportional to the logarithmic portfolio. Hence, the single-factor ICAPM holds, even though means, variances, and covariances may change stochastically over time.

The paper is organized as follows. Section 2 outlines the trading model. Section 3 defines the ICML and discusses consumption and portfolio strategies. Section 4 states and proves our two-fund separation theorem, while Section 5 discusses its interpretation and implications. In Section 6, we note that our result implies the instantaneous CAPM in equilibrium. We conclude in Section 7.

2 The Model

The notation follows Nielsen [13, 1999]. Securities prices, drifts, and Wiener processes are column vectors, while prices of risk, trading strategies, and portfolio strategies are row vectors. These conventions minimize the need to use transposes and lead to the cleanest possible notation.

We use a standard continuous-times model on a finite time horizon is $[0, T]$. Underlying the model is a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $F = (\mathcal{F}_t)_{t\in[0,T]}$ and a K-dimensional Wiener process W relative to F.

There are $N + 1$ basic long-lived securities, with price process

$$\bar{S} = \begin{pmatrix} M \\ S \end{pmatrix}$$

Security zero is a *money market account* with price process

$$M(t) = M(0) \exp\left\{\int_0^t r\, ds\right\}$$

where $M(0) > 0$ and r is the *interest rate process*, which will be assumed to be a deterministic function of time.

The prices of the remaining N securities are given by an N-dimensional vector S of Itô processes of the form

$$dS = \mathcal{D}(S)\mu\, dt + \mathcal{D}(S)\sigma\, dW$$

where $\mathcal{D}(S)$ is the diagonal matrix with the vector S along the diagonal. The process μ is N-dimensional and is assumed to be measurable, adapted, and pathwise locally integrable with respect to time. The process σ is $(N \times K)$-dimensional and is assumed to be measurable, adapted, and pathwise locally square integrable with respect to time. These processes are not assumed to be Markovian or to be functions of a vector of state variables.

Assume that σ has rank N. Set

$$\lambda = (\mu - r\iota)^\top \left(\sigma\sigma^\top\right)^{-1} \sigma$$

Then λ is a vector of *prices of risk*, which means that

$$\mu - r\iota = \sigma\lambda^\top$$

Assume that the process $\lambda\lambda^\top$ is locally integrable with respect to time. A *state price process* or *pricing kernel* with prices of risk λ is a positive one-dimensional Itô process Π such that

$$\frac{d\Pi}{\Pi} = -r\,dt - \lambda\,dW$$

Assume that the process

$$\eta = \frac{1}{\Pi(0)M(0)}\Pi M$$

is a martingale. Then there exists a *risk-adjusted probability measure* Q, which has density $\eta(T)$ with respect to P.

A *consumption flow process* is a measurable adapted process. If c is a consumption flow process which is integrable with respect to time with probability one, then the corresponding *cumulative consumption process* will be

$$C(t,\omega) = \int_0^t c(s)\,ds$$

and the normalized cumulative consumption process will be

$$C^{1/M}(t) = \int_0^t \frac{c(s)}{M(s)}\,ds$$

A *random final wealth level* is a random variable Y which is measurable with respect to $\mathcal{F}_T$.

The investor's object of choice is a pair (c, Y) of a consumption flow process and a random final wealth level. Such a pair will be said to be *admissible* if $Y/M(T)$ is integrable under Q and

$$E_Q \int_0^T \left|\frac{c(t)}{M(t)}\right| dt < \infty$$

If so, then $C^{1/M}(T)$ is also integrable under Q.

A *trading strategy* is an adapted and measurable process $\bar{\Delta} = (\Delta_0, \Delta)$ whose values are $(N+1)$-dimensional row vectors, such that the integral of $\bar{\Delta}\,d\bar{S}$ is well defined. The *value process* of $\bar{\Delta}$ is $\Delta_0 M + \Delta S$. A pair $(c, \bar{\Delta})$ of a consumption flow process c and a trading strategy $\bar{\Delta}$ will be called a *consumption and trading strategy*.

If $(c, \bar{\Delta})$ is a consumption and trading strategy such that c is integrable with respect to time with probability one, and such that for all t,

$$\bar{\Delta}(t)\bar{S}(t) + C(t) = \bar{\Delta}(0)\bar{S}(0) + \int_0^t \bar{\Delta}\, d\bar{S}$$

then $(\bar{\Delta}, c)$ is said to be *self-financing*. In normalized form,

$$\bar{\Delta}(t)\bar{S}(t)/M(t) + C^{1/M}(t) = \bar{\Delta}(0)\bar{S}(0)/M(0) + \int_0^t \bar{\Delta}\, d(\bar{S}/M)$$

Say that $(c, \bar{\Delta})$ is *admissible* if it is self-financing, $(c, \bar{\Delta}(T)\bar{S}(T))$ is admissible, and the process $\bar{\Delta}(t)\bar{S}(t)/M(t) + C^{1/M}(t)$ is a martingale under Q. This implies in particular that both $\bar{\Delta}(t)\bar{S}(t)/M(t)$ and $C^{1/M}(t)$ are integrable for every t.

Let I be an interval on the real line (typically the entire real line or $[0, \infty)$ or $(0, \infty)$). Let u and U be real functions on $I \times [0, T]$ and I, respectively, such that U is concave and $u(x, t)$ is concave for each $t \in [0, T]$. These are the investor's utility functions. Assume that

$$\int_0^T u(c(t), t)\, dt$$

and $U(Y)$ are integrable under P for every admissible pair (c, Y) of a consumption flow process and a random final wealth with values in I, such that

$$w_0 = M(0)E_Q(C^{1/M}(T) + Y/M(T))$$

This assumption essentially says that the investor cannot afford inifite expected utility.

3 Consumption Rate and Portfolio Strategies

A *consumption rate strategy* is an adapted measurable one dimensional process ξ. It tells us the fraction of wealth consumed at each instant. A *portfolio strategy* is an adapted measurable N-dimensional row vector valued process θ. The entries are the fractions of wealth invested in the various risky securities. They do not add up to one, because the remainder is invested in the money market account. A *consumption rate and portfolio strategy* is a pair (ξ, θ) of a consumption rate strategy ξ and a portfolio strategy θ.

If $(c, \bar{\Delta})$ is a self-financing consumption and trading strategy with $\bar{\Delta} = (\Delta_0, \Delta)$ such that the value process $V = \bar{\Delta}\bar{S}$ is positive, then the corresponding consumption rate and portfolio strategy (ξ, θ) are given by $\xi = c/V$ and $\theta = \Delta\mathcal{D}(S)/V$.

Conversely, if (ξ, θ) is a consumption rate and portfolio strategy and $w_0 > 0$ is an initial wealth level, then there is a unique self-financing consumption and trading strategy $(c, \bar{\Delta})$ such that $\bar{\Delta}(0)\bar{S}(0) = w_0$, $\bar{\Delta}\bar{S} > 0$, and (ξ, θ) is the portfolio and consumption rate strategy corresponding to $(c, \bar{\Delta})$. The value process $V = \bar{\Delta}\bar{S}$ of $\bar{\Delta}$ is the unique Itô process such that $V(0) = w_0$ and

$$\frac{dV}{V} = ((r + \theta(\mu - \iota r)\mu - \xi)\ dt + \theta\sigma\, dW$$

where ι is a column vector of ones, and $(c, \bar{\Delta})$ is given by $\xi = cV$, $\Delta = \mathcal{D}(S)^{-1}\theta V$, and $\Delta_0 M + \Delta S = V$.

Say that a consumption rate and portfolio strategy is *admissible* (given the initial wealth level w_0) if the corresponding consumption and trading strategy is admissible.

It follows that choosing an admissible consumption and trading strategy $(c, \bar{\Delta})$ such that the value process $V = \bar{\Delta}\bar{S}$ is positive, is equivalent to choosing the corresponding admissible portfolio and consumption rate strategy.

Recall from mean-variance theory that mean-variance efficient portfolios are portfolios that maximize the expected rate of return given the variance or standard deviation of the rate of return. We can similarly define *instantaneously efficient portfolios* as those that maximize the expected instantaneous rate of return given the standard deviation of the instantaneous rate of return. Their combinations of standard deviation of returns and expected returns plot on a straight line whose intercept with the expected-return axis is the instantaneous interest rate. We call this line the *instantaneous capital market line* (ICML).

It follows from the standard theory that the instantaneously efficient portfolios are the portfolios that are proportional to the portfolio $\phi^{\ln}$ given by

$$\phi^{\ln} = (\mu - r\iota)^\top \left(\sigma\sigma^\top\right)^{-1}$$

where we note that $\sigma\sigma^\top$ is the covariance matrix of the instantaneous rates of return to the various securities. We call this portfolio the *logarithmic portfolio* because, as is well known, it is indeed the optimal portfolio for an investor with a logarithmic utility function.

Observe that the vector of prices of risk λ is

$$\lambda = \phi^{\ln}\sigma$$

The slope of the instantaneous capital market line is the ratio of the excess expected instantaneous rate of return to the standard deviation of the instantaneous rate of return to the logarithmic portfolio. We can calculate this slope as follows.

The excess instantaneous expected rate of return to $\phi^{\ln}$ is

$$\phi^{\ln}(\mu - r\iota) = (\mu - r\iota)^\top \left(\sigma\sigma^\top\right)^{-1} (\mu - r\iota) = \lambda\lambda^\top$$

The variance of the instantaneous rate of return to $\phi^{\ln}$ is

$$\phi^{\ln}\sigma\sigma^\top\phi^{\ln\top} = \lambda\lambda^\top$$

and the standard deviation is $\sqrt{\lambda\lambda^\top}$. Hence, the slope of the ICML is

$$\frac{\phi^{\ln}(\mu - r\iota)}{\sqrt{\phi^{\ln}\sigma\sigma^\top\phi^{\ln\top}}} = \frac{\lambda\lambda^\top}{\sqrt{\lambda\lambda^\top}} = \sqrt{\lambda\lambda^\top}$$

It follows that the ICML is the straight line with intercept r and slope $\sqrt{\lambda\lambda^\top}$.

While the individual elements of the vector λ are prices of risk with respect to the individual Wiener processes, $\sqrt{\lambda\lambda^\top}$ is the price of risk in the aggregate. We can also think of it as the instantaneous Sharpe ratio for instantaneously mean-variance efficient portfolios.

4 Two-Fund Separation

The following theorem says that when the ICML is deterministic, investors prefer portfolios that are proportional to the logarithmic portfolio. Such portfolios plot on the ICML. If there exists an optimal consumption rate and portfolio strategy for an investor, then there exists one such that the portfolio strategy is proportional to the logarithmic portfolio and plots on the ICML.

By formulating the theorem in terms of dominance rather than optimality, we avoid having to impose assumptions to ensure the existence of an optimal consumption rate and portfolio strategy. This idea is borrowed from Khanna and Kulldorff [6, 1999] and Ross [16, 1978].

Theorem *Assume that the interest rate is deterministic and that the slope of the ICML is positive and deterministic. Then for each initial wealth level* $w_0 > 0$ *and each admissible consumption rate and portfolio strategy* (ξ, θ)*, there exists an admissible consumption rate and portfolio strategy* $(\tilde{\xi}, \tilde{\theta})$ *such that* $\tilde{\theta}$ *is proportional to the logarithmic portfolio, and such that every risk averter with initial wealth* w_0 *will prefer* $(\tilde{\xi}, \tilde{\theta})$ *to* (ξ, θ)*.*

The statement that $\tilde{\theta}$ is proportional to the logarithmic portfolio means that $\tilde{\Delta} = \alpha\phi^{\ln}$ for some one-dimensional measurable and adapted process α, which represents the value weight invested in the logarithmic portfolio. The remainder is invested in the money market account. It follows from the proof that in fact, α is adapted not only to the filtration F but to the coarser filtration F^{Π} generated by the state price process. It can be shown that α equals the relative risk tolerance of the investor's indirect utility function, as in Merton [9, 1973] for the case of constant moments.

Portfolios like $\tilde{\theta}$ that are proportional to the logarithmic portfolio plot on the ICML. Even with a constant ICML, the value weight α invested in the logarithmic portfolio will in general change over time in response to changes in the investor's wealth and resultant changes in the risk aversion of his indirect utility function. Changes in α can be interpreted as sliding up and down the instantaneous capital market line.

The remainder of this section is devoted to proving the theorem. The remaining sections discuss its interpretation and implications.

First, we need a version of the complete markets theorem for trading and consumption strategies.

Proposition *Assume that $F = F^W$ and that σ has rank K. Let (c, Y) be an admissible pair of a consumption flow process and a random final wealth level. Then there exists a trading strategy $\bar{\Theta}$ such that $(\bar{\Theta}, c)$ is an admissible trading and consumption strategy, and such that $\bar{\Theta}(T)\bar{S}(T) = Y$.*

PROOF: According to the usual complete markets theorem (Nielsen [13, 1999, Theorem 5.6], there exists a trading strategy $\bar{\Delta}$ such that the trading and consumption strategy $(\bar{\Delta}, 0)$ is admissible and such that $\bar{\Delta}(T)\bar{S}(T) = Y + M(T)C^{1/M}(T)$. Let $\bar{b} = (1, 0)$ be the trading strategy which consists in buying and holding one unit of the money market account. Then $\bar{b}\bar{S} = M$. Set $\bar{\Theta} = \bar{\Delta} - C^{1/M}\bar{b}$. Then

$$\bar{\Theta}(T)\bar{S}(T) = \bar{\Delta}(T)\bar{S}(T) - C^{1/M}(T)\bar{b}(T)\bar{S}(T) = Y$$

Since $(\bar{\Theta}, c) = (\bar{\Delta}, 0) + (-C^{1/M}\bar{b}, c)$, to show that $(\bar{\Theta}, c)$ is admissible it suffices to show that $(-C^{1/M}\bar{b}, c)$ is admissible. It is self-financing since $-C^{1/M}\bar{b}\bar{S}/M + C^{1/M} = 0$ and

$$-\int_0^t C^{1/M}\bar{b}\, d(\bar{S}/M) = 0$$

which follows from the fact that $\bar{b}\, d(\bar{S}/M) = 0$. It is admissible because the process

$$-C^{1/M}(t)\bar{b}(t)\bar{S}(t)/M(t) + C^{1/M}(t) = 0$$

is a martingale under Q. $\square$

Define a process B by

$$B(t) = \int_0^t \frac{1}{\sqrt{\lambda\lambda^\top}}\lambda\, dW$$

Then B is a Wiener process relative to the filtration F, and

$$\frac{d\Pi}{\Pi} = -r\, dt - \lambda\, dW = -r\, dt - \sqrt{\lambda\lambda^\top}\, dB$$

Since r and $\sqrt{\lambda\lambda^\top}$ are deterministic, Π is adapted to F^B.

Let $S^{\ln}$ be the value process of one unit of account initially invested in the logarithmic portfolio. Then

$$\begin{aligned}\frac{dS^{\ln}}{S^{\ln}} &= \Big((r + \phi^{\ln}(\mu - \iota r)\Big)\, dt + \phi^{\ln}\sigma\, dW \\ &= \Big(r + \lambda\lambda^{\top}\Big)\, dt + \lambda\, dW \\ &= \Big(r + \lambda\lambda^{\top}\Big)\, dt + \sqrt{\lambda\lambda^{\top}}\, dB\end{aligned}$$

Hence, $S^{\ln} = \Pi(0)/\Pi$, and $S^{\ln}$ is adapted to the filtration F^B.

Consider a simplified securities market model with one money market account and one instantaneously risky security, driven by one source of uncertainty, the one-dimensional Wiener process B on the filtration F^B. The instantaneously risky security has price process $S^{\ln}$, and the money market account is the same as in the original model.

It follows from the dynamics of $S^{\ln}$ above that the price of risk with respect to B is $\sqrt{\lambda\lambda^{\top}}$. Hence, the state price process in the simplified model is Π, the same as in the original model.

Let $(c, \bar{\Delta})$ be the self-financing and admissible trading and consumption strategy corresponding to (ξ, θ) with initial value w_0.

Pick a measurable adapted process $\tilde{c}$ such that for each t, $\tilde{c}(t) = E_Q(c(t) \mid \mathcal{F}_t^B)$. The existence of such a process follows from a combination of (a) the fact that F^B is both right-continuous and left-continuous (Liptser and Shiryayev [7, 1977, Theorem 4.8], (b) the predictable projection theorem (Jacod and Shiryaev [4, 1987, Chapter I, Theorem I.2.28], and (c) the fact that predictable processes are adapted and measurable (Nielsen [13, 1999, Proposition 1.42], Chung and Williams [2, 1990, p. 63]).

Simple calculations using interchange of the order of integration, repeated conditioning, and the conditional Jensen's inequality show that

$$E_Q \int_0^T \left|\frac{\tilde{c}}{M}\right| dt = E_Q \int_0^T \left|\frac{c}{M}\right| dt < \infty$$

Similar calculations show that the cumulative consumption process $\tilde{C}$ corresponding to $\tilde{c}$ satisfies

$$E_Q[\tilde{C}^{1/M}(T) - \tilde{C}^{1/M}(t) \mid \mathcal{F}_t^B] = E_Q[C^{1/M}(T) - C^{1/M}(t) \mid \mathcal{F}_t^B]$$

for each t.

Set $Y = \bar{\Delta}(T)\bar{S}(T)$. Then Y is positive and integrable with respect to Q. Set $\tilde{Y} = E_Q(Y \mid \mathcal{F}_T^B)$. Then $\tilde{Y}$ is positive, integrable with respect to Q, and measurable with respect to F_T^B, with $E_Q\tilde{Y} = E_Q Y$.

It follows that $(\tilde{c}, \tilde{Y})$ is an admissible pair of a consumption flow process and a random final wealth.

Since the simplified securities market model has filtration F^B and the single risky security has dispersion $\sqrt{\lambda\lambda^\top} > 0$, it follows from the proposition above that there exists a trading strategy $\bar{\Theta} = (\Theta_0, \Theta)$ in the simplified model such that $(\bar{\Theta}, \tilde{c})$ is a self-financing and admissible trading and consumption strategy, and such that $\bar{\Theta}(T)\bar{S}(T) = \tilde{Y}$.

Let V and $\tilde{V}$ be the value processes of $\bar{\Delta}$ and $\bar{\Theta}$: $V = \bar{\Delta}\bar{S}$ and $\tilde{V} = \bar{\Theta}\bar{S}$. Because $(\bar{\Delta}, c)$ and $(\bar{\Theta}, \tilde{c})$ are admissible, for each t,

$$V(t)/M(t) = E_Q(Y/M(T) + C^{1/M}(T) \mid \mathcal{F}_t) - C^{1/M}(t)$$

and

$$\tilde{V}(t)/M(t) = E_Q(\tilde{Y}/M(T) + \tilde{C}^{1/M}(T) \mid \mathcal{F}_t^B) - \tilde{C}^{1/M}(t)$$

Hence,

$$\begin{aligned}
&E_Q(V(t)/M(t) \mid \mathcal{F}_t^B) \\
&\quad = E_Q(Y/M(T) \mid \mathcal{F}_t^B) + E_Q(C^{1/M}(T) - C^{1/M}(t) \mid \mathcal{F}_t^B) \\
&\quad = E_Q(\tilde{Y}/M(T) \mid \mathcal{F}_t^B) + E_Q(\tilde{C}^{1/M}(T) - \tilde{C}^{1/M}(t) \mid \mathcal{F}_t^B) \\
&\quad = \tilde{V}(t)/M(t)
\end{aligned}$$

This implies that $\tilde{V}$ is positive. It also implies that the initial wealth required to implement the trading and consumption strategy $(\bar{\Theta}, \tilde{c})$ is w_0:

$$\tilde{V}(0) = M(0)E_Q(V(0)/M(0) \mid \mathcal{F}_0^B)] = M(0)w_0/M(0) = w_0$$

Now let $(\alpha, \tilde{\xi})$ be the portfolio and consumption rate strategy in the simplified model corresponding to $(\bar{\Theta}, \tilde{c})$: $\alpha = \Theta S^{\ln}/\tilde{V}$ and $\tilde{\xi} = \tilde{c}/\tilde{V}$. Given the initial wealth level w_0, the value process of $(\alpha, \tilde{\xi})$ is $\tilde{V}$.

Let $\tilde{\theta}$ be the portfolio strategy in the original model given by $\tilde{\theta} = \alpha\phi^{\ln}$. Then

$$\frac{d\tilde{V}}{\tilde{V}} = \left((r + \alpha\lambda\lambda^\top - \tilde{\xi}\right) dt + \alpha\sqrt{\lambda\lambda^\top}\, dB$$

$$\begin{aligned} &= \left((r + \alpha\phi^{\ln}(\mu - \iota r) - \tilde{\xi}\right) dt + \alpha\phi^{\ln}\sigma\, dW \\ &= \left((r + \tilde{\theta}(\mu - \iota r) - \tilde{\xi}\right) dt + \tilde{\theta}\sigma\, dW \end{aligned}$$

Hence, if $(\tilde{\theta}, \tilde{\xi})$ is started off at the initial wealth level w_0, then its value process is $\tilde{V}$. The corresponding pair of a consumption flow process and a random final wealth is $(\tilde{c}, \tilde{V}(T)) = (\tilde{c}, \tilde{Y})$.

Finally, it follows from two applications of the conditional Jensen's inequality that every risk averter will prefer $(\tilde{c}, \tilde{Y})$ to (c, Y):

$$EU(Y) = E[E(U(Y) \mid \mathcal{F}_T^B)] \leq EU(E(Y \mid \mathcal{F}_T^B)) = EU(Y^*)$$

and

$$\begin{aligned} E\int_0^T u(t, c(t))\, dt &= \int_0^T Eu(t, c(t))\, dt \\ &= \int_0^T E(E[u(t, c(t)) \mid \mathcal{F}_t^B])\, dt \\ &\leq \int_0^T Eu(t, E(c(t) \mid \mathcal{F}_t^B))\, dt \\ &= \int_0^T Eu(t, \tilde{c}(t)\, dt \\ &= E\int_0^T u(t, \tilde{c}(t))\, dt \end{aligned}$$

5 Interpretation and Implications

Apart from (1) the case of logarithmic utility, the only previously known results that we are aware of where all investors will simply invest in the money market account and the logarithmic portfolio are (2) Merton's [8, 1973] case where all moments are constant, (3) Constantinides' [3, 1980] Proposition 1 which assumes aggregating utility functions, equilibrium, and stationary returns to all assets in positive net supply, (4) an example in Karatzas et al. [5, 1991] which assumes power utility, complete markets, a deterministic interest rate r, and a deterministic vector λ of prices of risk; and (5) an example in Ocone and Karatzas [14, 1991], which also assumes complete markets, a deterministic interest rate r, and a deterministic vector λ of prices of risk.

The theorem generalizes all of these results, because it only assumes that the utility function exhibits risk aversion, and it allows σ, μ and λ to change randomly over time, so long as the relation

$$\mu - r\iota = \sigma\lambda^\top$$

is satisfied and r and $\lambda\lambda^\top$ remain deterministic.

If there are one thousand risky securities, then the case of constant moments requires the constancy of 501,500 parameters, because this is the number of free parameters in the instantaneous excess return vector $\mu - r\iota$ and the instantaneous covariance matrix $\sigma\sigma^\top$. The theorem imposes only a two-dimensional restriction on these parameters: the intercept and slope of the ICML should be constant (or equal to deterministic functions of time). In this sense, the theorem has 501,498 more degrees of freedom than the case of constant moments.

Example 1 Consider any process for the dispersion matrix σ. Let it incorporate any kind of time-varying conditional second moments that may be considered suitable or used in the empirical literature of conditional asset pricing, such as ARCH or GARCH. Let r and λ be any processes such that r and $\lambda\lambda^\top$ are deterministic but potentially time-dependent. Simply define the instantaneous mean return processes μ by

$$\mu = r\iota + \sigma\lambda^\top$$

Then the ICML is deterministic. □

The following is a more concrete, although admittedly contrived, example of what the process λ might look like.

Example 2 Suppose $N = K$. Suppose the interest rate $r > 0$ and the matrix σ are deterministic constants. Let $x \in \mathbb{R}^K$, $x \neq 0$, and suppose

$$\mu = r\iota + \frac{1}{\|W + x\|}\sigma(W + x)$$

Then

$$\lambda = \frac{1}{\|W + x\|}(W + x)^\top$$

and $\lambda\lambda^\top = 1$. According to the theorem, the optimal portfolio is proportional to the logarithmic portfolio. Although the instantaneous means vary over time in a highly random manner, investors will not hold hedge portfolios to hedge against these changes. Even so, the composition of their portfolio will change stochastically over time, since we find

$$\phi^{\ln} = \lambda\sigma^{-1} = \frac{1}{\|W + x\|}(W + x)^\top \sigma^{-1}$$

□

Merton [9, 1973], in the discrete-time introduction to his model, defined the "investment opportunity set" as the transition probabilities for returns on each asset over the next trading interval. In the continuous-time setting, he defined changes in the investment opportunity set as any changes in instantaneous means or variances or in the state variables that drive them (he assumed the correlations to be constant). As in Merton [8, 1973], he showed that if the investment opportunity set so defined is constant, then the investor will hold only the logarithmic portfolio and the money market account.

Merton's [11, 1990, Chapter 15, Section 10], [12, 1992] $(m + 2)$-fund separation theorem[1] says that if all the instantaneous means and variances of the securities returns are functions of a set of m state variables (the covariances of the securities returns are assumed to be constant), then investors will hold portfolios that are combinations of the money market account, the logarithmic portfolio, and m "hedge portfolios" that have maximal correlation with the m state variables and are interpreted as hedges against changes in the investment opportunity set.

Our theorem illuminates the separation result, since it underscores that it may not be necessary to hold any hedge portfolios at all even if the instantaneous means, variances, and covariances change randomly over time. Specifically, if these changes do not affect the instantaneous capital market line, then it is not necessary to hold any of the hedge portfolios.

[1] This theorem was added in the book but was not there in the original article, Merton [9, 1973].

6 The Single-Factor ICAPM

Under the assumptions of the theorem, all investors will hold portfolios proportional to the logarithmic portfolio. Therefore, in equilibrium, the market portfolio will be proportional to the logarithmic portfolio. Hence, the single-factor ICAPM holds, with the single factor being the return on the market portfolio:

$$\mu_i - r = \beta_i(\mu_m - r)$$

Here, $\beta_i = e_i \sigma \sigma^\top e^{i\top} / (e_i \sigma \sigma^\top m^\top)$ is the beta of security i with respect to the market portfolio, e_i is the portfolio strategy that consists in holding all wealth in security i, m is the market portfolio, and $\mu_m = m\mu$ is the instantaneous rate of return on the market portfolio.

Merton [9, 1973] derived the single-factor ICAPM under the assumption that the interest rate and the instantaneous variances and covariances of securities returns are deterministic. Here we assume only that the ICML is deterministic.

The consumption CAPM of Breeden [1, 1979] is also a single factor model which allows for stochastically changing means, variances and covariances; however, the single factor is changes in aggregate consumption rather than the return on the market portfolio.

In the discrete-time Sharpe-Lintner CAPM, if the interest rate, and the means, variances and covariances of securities returns are constant, then the market portfolio has to be constant, which means that its value weights are constant. In the absence of random changes in the supplies of the securities, this implies that all the prices must always change proportionally, and hence, they must be perfectly correlated. This is of course inconsistent both with the model and with reality. This was first observed by Rosenberg and Ohlson [15, 1976].

Our version of the ICAPM resolves the Rosenberg-Ohlson paradox. Since it allows the means, variances and covariances to change over time, the value weights of the market portfolio will change even though the capital market line stays constant. The instantaneous price changes of the various securities will not be proportional to each other.

7 Conclusions

This study has shown that if the intercept and slope of what we call the instantaneous capital market line (ICML) are deterministic, then investors will not hedge against changes in the means, variances or covariances of security returns. They will simply hold a possibly time-varying combination of two funds that span the ICML: the riskless asset and the logarithmic portfolio. Investors will place themselves along the ICML, and they will slide up and down the ICML over time as their wealth and risk tolerance change.

It is common in the empirical asset pricing literature to allow for randomly time-varying moments of the returns to securities or to the market portfolio, while ignoring the intertemporal hedging premia that should be present in the specification according to Merton [9, 1973]. Our analysis provides a theoretical framework for some of these tests. A constant or deterministic ICML implies that the intertemporal hedging premia disappear. Therefore, they can be ignored in empirical tests whose specification is consistent with a constant or deterministic ICML.

We derived a single-factor ICAPM with time-varying means, variances, and covariances. Our model is consistent with one of the empirical specifications proposed in Merton [10, 1980] for estimating the expected return to the market, and with ARCH-M and GARCH-M specifications that have been used in the empirical literature. It is different from the consumption CAPM of Breeden [1, 1979] because the single factor is the return to the market portfolio rather than changes in aggregate consumption. Our version of the ICAPM resolves the paradox of Rosenberg and Ohlson [15, 1976].

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