Monotone Risk Aversion[1]

Lars Tyge Nielsen[2]

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[2]Morgan Stanley, 750 7th Avenue, New York, NY 10019, tel: 212 762 2998, e-mail: lars.nielsen@morganstanley.com

Abstract

This paper defines decreasing absolute risk aversion in purely behavioral terms without any assumption of differentiability and shows that a strictly increasing and risk averse utility function with decreasing absolute risk aversion is necessarily differentiable with an absolutely continuous derivative. A risk averse utility function has decreasing absolute risk aversion if and only if it has a decreasing absolute risk aversion density, and if and only if the cumulative absolute risk aversion function is increasing and concave. This leads to a characterization of all such utility functions. Analogues of these results also hold for increasing absolute and for increasing and decreasing relative risk aversion.

1 Introduction

Decreasing absolute risk aversion means that the decision maker behaves in a less risk averse fashion the larger his wealth. Pratt [7, 1964] defined it by the condition that the equivalent risk premium $\pi(x, z)$ should be a decreasing function of initial wealth x, for every random addition z to wealth. Dybvig and Lippman [6, 1983], following Yaari [12, 1969], defined decreasing risk aversion by requiring that "gambles accepted at a given level of wealth will be accepted at all higher levels of wealth." These definitions are equivalent, and they are behavioral in the sense that they can be directly interpreted in terms of preferences or choice behavior.

An important fact about decreasing absolute risk aversion is that it can be expressed in terms of the Arrow-Pratt coefficient of absolute risk aversion. Indeed, a utility function exhibits decreasing absolute risk aversion if and only if the coefficient of absolute risk aversion is decreasing. This was shown by Pratt [7, 1964] Pratt and by Dybvig and Lippman [6, 1983]. Arrow [1, 1965] stated this result as well, except that his terminology was inverted relative to Pratt's. Arrow used the decreasing coefficient of absolute risk aversion as a definition of decreasing absolute risk aversion, and he then argued that it is equivalent to more risk averse behavior at higher wealth levels.

The definition of the coefficient of absolute risk aversion assumes that the utility function is twice differentiable. Pratt's proof that one utility function is more risk averse than another if and only if it globally has a larger coefficient of risk aversion assumes that the utility functions are twice continuously differentiable. The same assumption therefore underlies the proof that decreasing absolute risk aversion is equivalent to a decreasing coefficient of absolute risk aversion. Since Dybvig and Lippman use Pratt's results, their proof also relies on this assumption.

However, a utility function with decreasing absolute risk aversion is not necessarily twice continuously differentiable If it is strictly increasing, then it has to be everywhere differentiable with positive derivative. If, in addition, it is concave and differentiable, then it is automatically continuously differentiable; but the second derivative may not exist everywhere.

This paper defines the concept of an absolute risk aversion density, which is a generalization of the coefficient of absolute risk aversion. The paper's main result, Theorem 1, says that a strictly increasing risk averse utility function has decreasing absolute risk aversion if and only if it has a decreasing absolute risk aversion density. It generalizes the result of Pratt [7, 1964], Arrow [1, 1965], and Dybvig and Lippman [6, 1983] to utility functions that are not assumed to be twice continuously differentiable.

Specifically, if the utility function is strictly increasing and risk averse, and if it exhibits decreasing absolut risk averison, then the marginal utility function is absolutely continuous, which means that it has a density function. Since the marginal utility function is absolutely continuous, the same is true of what we call the cumulative absolute risk aversion function, the negative of the logarithm of the marginal utility. This allows us to define the absolute risk aversion density as the density of the cumulative absolute risk aversion function. It equals the negative of the density of the marginal utility divided by the marginal utility itself, and it coincides with the usual coefficient of absolute risk aversion whenever the latter is defined. It is unique almost everywhere.

An idea similar to the absolute risk aversion density appeared in Vickson [9, 1975], [10, 1975], [11, 1977], where a DARA utility function was defined as one which has, in our terminology, a non-negative, decreasing, and "piece-wise smooth" absolute risk aversion density.

Theorem 1 also says that a strictly increasing risk averse utility function has decreasing absolute risk aversion if and only if its cumulative absolute risk aversion function is increasing and concave. This allows a complete characterization of strictly increasing risk averse utility functions with decreasing absolute risk aversion. The characterization says that such a utility function is uniquely determined by its cumulative absolute risk aversion function, which can be any increasing concave function, and an additive constant. This is so because the utility function can be recovered, up to an additive constant, by integrating the exponential of the cumulative absolute risk aversion function.

Pratt [7, 1964] wrote that "..., convenient utility functions for which [the coefficient of absolute risk aversion] is decreasing are not so very easy to find." Our characterization represents a way of finding all such utility functions.

All the results hold for increasing absolute risk aversion as well, except that the cumulative absolute risk aversion function will be convex rather than concave, and the absolute risk aversion density will be increasing rather than decreasing. Thus, a strictly increasing risk averse utility function has increasing absolute risk aversion if and only if its cumulative absolute risk aversion function is increasing and convex, and if and only if it has an increasing absolute risk aversion density.

Analogues of all the results also hold for relative risk aversion, where increasing or decreasing relative risk aversion is defined in behavioral terms without assuming that the utility function is twice differentiable.

The cumulative relative risk aversion function of a utility function defined on the positive half-line is defined as the composition of the exponential function and the cumulative absolute risk aversion function. If it is absolutely continuous, then we call its density the relative risk aversion density.

If the utility function is strictly increasing and exhibits increasing or decreasing relative risk aversion, then it is necessarily differentiable with positive derivative, so that the cumulative absolute and the cumulative relative risk aversion functions are well defined. If, in addition, the utility function is risk averse, then the marginal utility functions, the cumulative absolute and the cumulative relative risk aversion functions are all absolutely continuous.

Theorem 2 says that a strictly increasing risk averse utility function has increasing (decreasing) relative risk aversion if and only if it has an increasing (decreasing) relative risk aversion density. The theorem also says that a strictly increasing risk averse utility function has increasing (decreasing) relative risk aversion if and only if its cumulative relative risk aversion function is increasing and convex (concave). This implies that a strictly increasing risk averse utility function with increasing (decreasing) absolute risk aversion is uniquely determined by its cumulative absolute risk aversion function, which can be any increasing and convex (concave) function, and an additive constant.

2 Monotone Absolute Risk Aversion

We consider a decision model where the relevant set of outcomes is an open interval I on the real line, unbounded above. The outcomes may be interpreted as levels of future wealth or future consumption. Typically, $I = \mathbb{R}$ or $I = (0, \infty)$.

The utility function u will be assumed to be strictly increasing, reflecting the idea that more consumption or more wealth is better.

A random variable z will be called a *binary lottery* if it has at most two distinct values.

Recall that the utility function u is said to be *risk averse* if

$$Eu(x + z) \leq u(x)$$

whenever x is in I and z is a binary lottery with $Ez \leq 0$ and such that $x + z \in I$ with probability one. It is well known that u is risk averse if and only if it is concave.

Recall that utility function u is *more risk averse* than a utility function v if

$$Eu(x + z) \leq u(x)$$

whenever x is in I and z is a binary lottery such that $x+z \in I$ with probability one and such that $Ev(x + z) \leq v(x)$.

We say that u exhibits decreasing absolute risk aversion (respectively, increasing absolute risk aversion) if it is less risk averse (respectively, more risk averse) at higher wealth levels. Decreasing or increasing absolute risk aversion are referred to jointly as monotone absolute risk aversion.

To express this formally, for every $d \geq 0$, define a utility function $u_{[d]}$ on I by

$$u_{[d]}(x) = u(x + d)$$

Say that u exhibits

- *decreasing absolute risk aversion* if for all $d \geq 0$, u is more risk averse than $u_{[d]}$ on I

- *increasing absolute risk aversion* if for all $d \geq 0$, $u_{[d]}$ is more risk averse than u on I
- *constant absolute risk aversion* if u exhibits both increasing and decreasing absolute risk aversion.

According to Proposition 1, a strictly increasing utility function with increasing or decreasing absolute risk aversion must be differentiable. The intuition is that kinks are not possible, because somehow the utility function would be infinitely risk averse at the kink. It would then have to be infinitely risk averse at any higher wealth level also, in the case of decreasing absolute risk aversion, or at any lower wealth level, in the case of increasing absolute risk aversion. This is not possible.

In the case of increasing absolute risk aversion, the result in Proposition 1 is known from Lemma 8.4.7 of Dubins and Savage [4, 1965], [5, 1976]. We provide a proof of Proposition 1 in the appendix.

Proposition 1 *Let u be a strictly increasing utility function on I. Suppose u exhibits decreasing or increasing absolute risk aversion. Then u is differentiable with $u' > 0$.*

Proposition 2 is a generalization of a part of the proof of Pratt [7, 1964, Theorem 1]. Pratt assumed the utility functions to be twice continuously differentiable.

Proposition 2 *Let u and v be differentiable utility functions on an interval I with $u' > 0$ and $v' > 0$. Then u is more risk averse than v if and only if u'/v' is continuous and*

$$\ln u'(r) - \ln u'(s) \geq \ln v'(r) - \ln v'(s)$$

whenever $r, s \in I$, $r < s$.

PROOF:

Let $k : v(I) \rightarrow I\!R$ be the function such that $u = k \circ v$. It is strictly increasing. Since $k = v^{-1} \circ u$, k is differentiable, and

$$u'(x) = k'(v(x))v'(x)$$

for $x \in I$.

Now u is more risk averse than v if and only if k is concave, which is the case if and only if k' is continuous and decreasing. But k' is continuous and decreasing if and only if the function u'/v' is continuous and decreasing. Take logs and rearrange to see that u'/v' is decreasing if and only if

$$\ln u'(r) - \ln u'(s) \geq \ln v'(r) - \ln v'(s)$$

whenever $r, s \in I$, $r < s$.

$\square$

Because of Proposition 2, we call $-\ln u'$ the *cumulative absolute risk aversion function* of u, provided that u is differentiable with positive derivative.

If u the cumulative absolute risk aversion function exists (u is differentiable with positive derivative), then u is risk averse (concave) if and only if the cumulative absolute risk aversion function is increasing.

Recall that a function $G : I \rightarrow I\!R$ is *absolutely continuous* if and only if there exists a measurable function $g : I \rightarrow I\!R$ such that

$$\int_s^r |g(t)|\, dt < \infty$$

and

$$G(r) - G(s) = \int_s^r g(t)\, dt$$

for $r, s \in I$. If so, then we may call g a *density* of G. A density of G is unique almost everywhere.

If u is differentiable with $u' > 0$ and the cumulative absolute risk aversion function $-\ln \circ u'$ is absolutely continuous, then a density $R_A^*(u)$ of $-\ln \circ u'$ will be called an *absolute risk aversion density* for u. If u has an absolute risk aversion density $R_A^*(u)$, then $R_A^*(u)$ is unique almost everywhere.

Corollary 1 *Let u and v be differentiable utility functions on an interval I with $u' > 0$ and $v' > 0$. Assume that $R_A^*(u)$ and $R_A^*(v)$ exist. Then u is more risk averse than v if and only if*

$$R_A^*(u) \geq R_A^*(v)$$

almost everywhere.

PROOF:

It follows from Proposition 2 that u is more risk averse than v if and only if

$$\int_r^s R_A^*(u)(t)\,dt \geq \int_r^s R_A^*(u)(t)\,dt$$

whenever $r, s \in I$, $r < s$. But this holds if and only if

$$R_A^*(u) \geq R_A^*(v)$$

almost everywhere.

□

If $R_A^*(u)$ exists, then $-\ln u'$ is differentiable at almost every x, with

$$(-\ln u')'(x) = R_A^*(u)(x)$$

and, hence,

$$R_A^*(u)(x) = -\frac{u''(x)}{u'(x)}$$

at almost every x. So, the usual coefficient of absolute risk aversion is almost everywhere well defined and equal to $R_A^*(u)$. Conversely, if u is twice continuously differentiable, then $R_A^*(u)$ exists and is equal to the usual coefficient of absolute risk aversion almost everywhere.

Proposition 3 *Provided that u is differentiable with $u' > 0$, $-\ln \circ u'$ is absolutely continuous if and only if u' is absolutely continuous; and $R_A^*(u)$ is an absolute risk aversion density of u if and only if $-u' R_A^*(u)$ is a density of u'.*

PROOF:

Suppose $R_A^*(u)$ is an absolute risk aversion density of u. Then

$$\ln \circ u'(s) = \ln \circ u'(r) - \int_r^s R_A^*(u)(t)\, dt$$

Apply the exponential function to the function $\ln \circ u'$. By a version of the chain rule for an absolutely continuous function (or Itô's Lemma applied to the deterministic Itô process $\ln u'$, interpreting s as time),

$$\begin{aligned} u'(s) &= \exp(\ln \circ u'(s)) \\ &= \exp(\ln \circ u'(r)) - \int_r^s \exp(\ln \circ u'(t)) R_A^*(u)(t)\, dt \\ &= u'(r) - \int_r^s u'(t) R_A^*(u)(t)\, dt \end{aligned}$$

so that $-u' R_A^*(u)$ is a density of u'. Conversely, if $-u' R_A^*(u)$ is a density of u', then a similar calculation using the logarithmic transformation shows that $R_A^*(u)$ is a density of $- \ln \circ u'$.

□

Theorem 1 *Let u be a strictly increasing risk averse utility function on I. The following statements are equivalent:*

1. *u exhibits decreasing (increasing) absolute risk aversion*
2. *u is differentiable with $u' > 0$, and the cumulative absolute risk aversion function*

   $$x \mapsto - \ln u'(x)$$

   is concave (convex)
3. *u has an absolute risk aversion density $R_A^*(u)$ which is decreasing (increasing).*

In the case of increasing absolute risk aversion, the equivalence of (1) and (2) in Proposition 1 is from Lemma 8.4.2 of Dubins and Savage [4, 1965] [5, 1976]. Pratt [7, 1964] and Dybvig and Lippman [6, 1983] proved the equivalence of (1) and (3) under the assumption that u is twice continuously differentiable.

Theorem 1 implies that if a strictly increasing risk averse utility function u exhibits monotone (decreasing or increasing) absolute risk aversion, then it is not only differentiable but continuously differentiable. It is twice differentiable everywhere except possibly at a countable number of points. If it is twice differentiable everywhere, then it is twice continuously differentiable. All this follows from the fact that the cumulative absolute risk aversion function $-\ln \circ u'$ is concave or convex. A concave or convex function is differentiable everywhere except possibly at a countable number of points, and if it is differentiable everywhere, then it is continuously differentiable.

It follows from the theorem that a strictly increasing utility function u is risk averse and exhibits decreasing (increasing) absolute risk aversion if and only if it has the form

$$u(s) = u(r) + \int_r^s \exp(-k(x))\, dx$$

for $r, s \in I$, for some increasing concave (convex) function k (the cumulative absolute risk aversion function of u). If we fix $r \in I$ and set $u(r) = 0$, then this equation establishes a bijection between increasing concave (convex) functions k and strictly increasing risk averse utility functions u that exhibit decreasing (increasing) absolute risk aversion and have value zero at r.

Pratt [7, 1964] wrote that "..., convenient utility functions for which [the coefficient of absolute risk aversion] is decreasing are not so very easy to find." The equation above represents a way of finding all such utility functions.

Theorem 1 assumes that the utility function is risk averse. This assumption is perfectly reasonable from an economic point of view, but it is somewhat stronger than what is actually needed. What is needed is to know, when proving that (1) implies (2), that the derivative u' cannot be discontinuous at every point. Risk aversion implies that u' is discontinuous at most at countably many points. This remark will be important when proving Theorem 2 on the basis of Theorem 1.

The rest of this section is devoted to the proof of Theorem 1.

Proof of Theorem 1:

It follows directly from Proposition 2 and Proposition 1 that (1) of the theorem holds if and only if u is differentiable with $u' > 0$ and the

$$x \mapsto \ln u'(x) - \ln u'(x+h)$$

is continuous and decreasing (increasing) for all $h \geq 0$.

(1) implies (2):

First, we observe that u' must be continuous. This is seen as follows. Suppose there is a point $x \in I$ where u' is discontinuous. Then $\ln \circ u'$ is discontinuous at x, and since $x \mapsto \ln u'(x) - \ln u'(x+h)$ is continuous for all $h \geq 0$, u' is discontinuous at $x+h$ for all $h \geq 0$. However, since u is risk averse, u' is decreasing, which implies that it is continuous (in fact, differentiable) almost everywhere. This is a contradiction, so u' is indeed continuous.

Let $n \in \mathbb{N}$, set $h = 2^{-n}$, and set

$$\mathcal{D}_n = \left\{ \frac{a}{2^n} \in I : a \in \mathcal{Z} \right\}$$

The function

$$x \mapsto \ln u'(x) - \ln u'(x+h)$$

is decreasing on

$$\{x \in \mathcal{D}_n : x + h \in \mathcal{D}_n\}$$

Hence, the function $-\ln \circ u'$ is concave (convex) on $\mathcal{D}_n$. It follows that it is concave (convex) on the set

$$\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}_n = \left\{ \frac{a}{2^n} \in I : n \in \mathbb{N} \text{ and } a \in \mathcal{Z} \right\}$$

Since $\mathcal{D}$ is dense in I and $-\ln u'$ is continuous, this function is concave (convex) on I.

(2) implies (3):

Since $-\ln \circ u'$ is concave (convex), it is absolutely continuous with density equal to its right derivative $(-\ln \circ u')'_+$ from the right, as shown by Rockafellar [8, 1970, Corollary 24.2.1]). So, u has the absolute risk aversion density $R_R^*(u) = (-\ln \circ u')'_+$, which is decreasing (increasing).

(3) implies (1):

Since

$$\ln u'(x) - \ln u'(x+h) = \int_x^{x+h} R_A^*(u)(t)\, dt$$

u is differentiable with $u' > 0$, and $R_A^*(u)$ is decreasing (increasing), the function

$$x \mapsto \ln u'(x) - \ln u'(x+h)$$

is continuous and decreasing (increasing) for all $h \geq 0$.

□

3 Monotone Relative Risk Aversion

Let $\hat{u}$ be a strictly increasing utility function on $(0, \infty)$.

We say that $\hat{u}$ exhibits decreasing relative risk aversion (respectively, increasing relative risk aversion) if it implies less risk averse (respectively, more risk averse) behavior at higher wealth levels, in the sense both initial wealth and the random additions to wealth under consideration are scaled up by the same factor of proportionality. Decreasing or increasing relative risk aversion are referred to jointly as monotone relative risk aversion.

To express this formally, for every $\hat{d} > 0$, define a utility function $\hat{u}_{\{\hat{d}\}}$ on $(0, \infty)$ by

$$\hat{u}_{\{\hat{d}\}}(\hat{x}) = \hat{u}(\hat{d}\hat{x})$$

Say that $\hat{u}$ exhibits

- *decreasing relative risk aversion* if for all $\hat{d} \geq 1$, $\hat{u}$ is more risk averse than $\hat{u}_{\{\hat{d}\}}$
- *increasing relative risk aversion* if for all $\hat{d} \geq 1$, $\hat{u}_{\{\hat{d}\}}$ is more risk averse than $\hat{u}$
- *constant relative risk aversion* if $\hat{u}$ exhibits both increasing and decreasing relative risk aversion.

The logarithmic transformation maps relative risk aversion into absolute risk aversion, in the following sense.

Let u be the utility function defined on $I\!R$ by $u = \hat{u} \circ \exp$. Then $\hat{u} = u \circ \ln$. If $\hat{x} > 0$ and $\hat{d} \geq 1$, set $x = \ln \hat{x}$ and $d = \ln \hat{d}$. Then $d \geq 0$, and it is easily verified that

$$u_{[d]} \circ \ln = \hat{u}_{\{\hat{d}\}}$$

Consequently, $\hat{u}$ exhibits increasing (respectively, decreasing, constant) relative risk aversion if and only if u exhibits increasing (respectively, decreasing, constant) relative risk aversion.

Proposition 4 *Let $\hat{u}$ be a strictly increasing utility function on $(0, \infty)$. Suppose $\hat{u}$ exhibits increasing or decreasing relative risk aversion. Then $\hat{u}$ is differentiable with $\hat{u}' > 0$.*

Proof: Set $u = \hat{u} \circ \exp$. Then u exhibits increasing or decreasing absolute risk aversion on $I\!R$, and hence, by Proposition 1, it is differentiable with $u' > 0$. But then $\hat{u} = u \circ \ln$ is differentiable with $\hat{u}'(\hat{x}) = u'(\ln \hat{x})/\hat{x} > 0$. □

If $\hat{u}$ is a differentiable utility function on $(0, \infty)$ with $\hat{u}' > 0$, then the *cumulative relative risk aversion function* of $\hat{u}$ is the function $-\ln \circ \hat{u}' \circ \exp$, defined on $I\!R$.

If $\hat{u}$ is differentiable with $\hat{u}' > 0$, and if the cumulative relative risk aversion function $-\ln \circ \hat{u}' \circ \exp$ is absolutely continuous, then a *relative risk aversion density* for $\hat{u}$ is a function $R_R^*(\hat{u})$, defined on $(0, \infty)$, such that $R_R^*(\hat{u}) \circ \exp$ is a density of $-\ln \circ \hat{u}' \circ \exp$. In other words,

$$\int_r^s |R_R^*(\hat{u})(\exp x)|\, dx < 0$$

and

$$\ln \circ \hat{u}' \circ \exp(r) - \ln \circ \hat{u}' \circ \exp(s) = \int_r^s R_R^*(\hat{u})(\exp x)\, dx$$

for all $r, s \in I\!R$.

If $\hat{u}$ has a relative risk aversion density $R_R^*(\hat{u})$, then $R_R^*(\hat{u})$ is unique almost everywhere.

Provided that $\hat{u}$ is differentiable with $\hat{u}' > 0$, the cumulative relative risk aversion function $-\ln \circ \hat{u}' \circ \exp$ is absolutely continuous if and only if the cumulative absolute risk aversion function $-\ln \circ \hat{u}'$ is absolutely continuous; and $R_R^*(\hat{u})$ is a relative risk aversion density of $\hat{u}$ if and only if it is a density of $\hat{u}'$. This follows from the identity,

$$\int_r^s R_R^*(\hat{u})(\exp x)\ dx = \int_{\exp r}^{\exp s} R_R^*(\hat{u})(y)\, dy$$

If $R_R^*(\hat{u})$ exists, then $-\ln \hat{u}' \circ \exp$ is differentiable at almost every x, with

$$(-\ln \circ \hat{u}' \circ \exp)'(x) = R_R^*(\hat{u})(\exp x)$$

and, hence,

$$R_R^*(\hat{u})(\hat{x}) = -\frac{\hat{x}\hat{u}''(\hat{x})}{\hat{u}'(\hat{x})}$$

at almost every $\hat{x} > 0$. So, the usual coefficient of relative risk aversion is almost everywhere well defined and equal to $R_R^*(\hat{u})$. Conversely, if $\hat{u}$ is twice continuously differentiable, then $R_R^*(\hat{u})$ exists and is equal to the usual coefficient of relative risk aversion almost everywhere.

If $\hat{u}$ is a differentiable utility function defined on $(0, \infty)$, with $\hat{u}' > 0$, set $u = \hat{u} \circ \exp$. Then the cumulative absolute risk aversion function of u and the cumulative relative risk aversion function of $\hat{u}$ are related by

$$-\ln \circ u'(x) = -\ln \circ \hat{u}' \circ \exp(x) - x$$

Proposition 5 *Let $\hat{u}$ be a differentiable utility function defined on $(0, \infty)$ with $\hat{u}' > 0$, and set $u = \hat{u} \circ \exp$. Then the cumulative relative risk aversion function of $\hat{u}$ is absolutely continuous if and only if the cumulative absolute risk aversion function of u is absolutely continuous; and $R_R^*(\hat{u})$ is a relative risk aversion density of $\hat{u}$ if and only if $R_A^*(u) = R_R^*(\hat{u}) \circ \exp - 1$ is an absolute risk aversion density of u.*

Proof: The functions $R_A^*(u)$ and $R_R^*(\hat{u})$ are related by the equation in the proposition if and only if

$$\int_r^s R_A^*(u)(x)\, dx = \int_r^s R_R^*(\hat{u})(\exp x)\, dx - (s - r)$$

for all $r, s \in I$. Since

$$-\ln \circ u'(x) = -\ln \circ \hat{u}' \circ \exp(x) - x$$

it follows that $R_A^*(u)$ is a density of $-\ln \circ u'$ if and only if $R_R^*(u) \circ \exp$ is a density of $-\ln \circ \hat{u}'$. □

Theorem 2 *Let $\hat{u}$ be a strictly increasing risk averse utility function on $(0, \infty)$ The following statements are equivalent:*

1. *$\hat{u}$ exhibits decreasing (increasing) relative risk aversion*
2. *$\hat{u}$ is differentiable with $\hat{u}' > 0$, and the cumulative relative risk aversion function*

   $$x \mapsto -\ln \circ \hat{u}' \circ \exp(x)$$

   is concave (convex)
3. *$\hat{u}$ has a relative risk aversion density $R_R^*(\hat{u})$ which is decreasing (increasing).*

PROOF:

The theorem follows from Theorem 1 applied to the function $u = \hat{u} \circ \exp$. Note that u is not necessarily risk averse, as assumed in Theorem 1. This does not matter, because $\hat{u}'$ and hence u' will be continuous almost everywhere, which is all that is needed in the proof of Theorem 1.

□

Theorem 2 implies that if a strictly increasing risk averse utility function $\hat{u}$ exhibits monotone (decreasing or increasing) relative risk aversion, then it is not only differentiable but continuously differentiable. It is twice differentiable everywhere except possibly at a countable number of points. If it is twice differentiable everywhere, then it is twice continuously differentiable. All this follows from the fact that the cumulative relative risk aversion function $-\ln \circ \hat{u}' \circ \exp$ is concave or convex. A concave or convex function is differentiable everywhere except possibly at a countable number of points, and if it is differentiable everywhere, then it is continuously differentiable.

It follows from the theorem that a strictly increasing utility function $\hat{u}$ on $(0,\infty)$ is risk averse and exhibits decreasing (increasing) relative risk aversion if and only if it has the form

$$\hat{u}(\hat{s}) = \hat{u}(\hat{r}) + \int_{\hat{r}}^{\hat{s}} \exp(-h \circ \ln(\hat{x}))\, d\hat{x}$$

for $\hat{r}, \hat{s} \in (0,\infty)$, for some increasing concave (convex) function h on $I\!R$ (the cumulative relative risk aversion function of $\hat{u}$). If we fix $\hat{r} > 0$ and set $\hat{u}(\hat{r}) = 0$, then this equation establishes a bijection between increasing concave (convex) functions h and strictly increasing risk averse utility functions that exhibit decreasing (increasing) relative risk aversion and have value zero at $\hat{r}$.

4 Appendix: Proof of Proposition 1

The proof of Proposition 1 relies on the following lemma.

To state the lemma, we need to be careful in defining what we mean by a concave function on a subset of the real line which is not necessarily an interval. Generally, if $C \subset I\!R$ is a set which is not necessarily an interval, and if k is a real valued function defined on C, then we say that k is *concave* if

$$k(tx + (1-t)y) \geq tk(x) + (1-t)k(y)$$

whenever $0 \leq t \leq 1$ *and all the points* x, y *and* $tx + (1-t)y$ *are in* C.

Lemma 1 *Let $C \subset I\!R$ be a set which has no smallest and no largest element, and let $h : C \to I\!R$ and $k : h(C) \to I\!R$ be strictly increasing concave functions.*

1. *If $k \circ h$ is continuous at a point x in C then so is h*
2. *If C and $h(C)$ are intervals, and if $k \circ h$ is differentiable at a point x in C then so is h*

PROOF:

(1):

Assume, to the contrary, that $k \circ h$ is continuous at x but h is discontinuous at x. Set

$$\underline{h} = \sup\{h(y) : y \in C, y < x\}$$
$$\overline{h} = \inf\{h(y) : y \in C, y > x\}$$
$$\underline{k} = \sup\{k(z) : z \in h(C), z < h(x)\} = \sup\{k \circ h(y) : y \in C, y < x\}$$

and

$$\overline{k} = \inf\{k(z) : z \in h(C), z > h(x)\} = \inf\{k \circ h(y) : y \in C, y > x\}$$

Then $\underline{k} = \overline{k}$ but $\underline{h} < \overline{h}$. Let $z, z' \in h(C)$ be such that $z < \underline{h}$ and $\overline{h} < z'$. Because k is concave, the graph of k above z' is on or below the line through $(z, k(z))$ and $(z', k(z'))$. Since this is true of all such z and z', the graph of k above $(\overline{h}, \overline{k})$ is on or below the line through $(\underline{h}, \underline{k})$ and $(\overline{h}, \overline{k})$. But since $\underline{k} = \overline{k}$, this line is horizontal. The point $(z', k(z'))$ on the graph of k approaches the point $(\overline{h}, \overline{k})$ as $z' \to \overline{h}$ from above through $h(C)$. But then, since k is strictly increasing, its graph cannot possibly stay on or below the horizontal line through $(\underline{h}, \underline{k})$ and $(\overline{h}, \overline{k})$, a contradiction.

(2):

If h has a kink at x, then k can only make it worse, so $k \circ h$ will also have a kink at x.

□

It is easily seen that if u and v are strictly increasing utility functions defined on an interval I, then u is more risk averse than v if and only if there exists a strictly increasing concave function $k : v(I) \to \mathbb{R}$ such that $u = k \circ v$. This is true even if v is not continuous, so that the set $v(I)$ where k is defined is not an interval.

Proof of Proposition 1:

Assume that u exhibits increasing absolute risk aversion.

Since u is strictly increasing, it is almost everywhere differentiable, by Billingsley [3, 1986, Theorem 31.2]). In particular, it is continuous at almost every point in I.

First, we show that u is continuous. Let x be a point in I where u is continuous. We will show that u is continuous at $x + d$ for every $d \geq 0$. It then follows that u is continuous everywhere in I.

Given $d \geq 0$, since $u_{[d]}$ is more risk averse than u, there is a strictly increasing concave function $k_d : u(I) \to I\!R$ such that $u_{[d]} = k_d \circ u$ on $u(I)$.

We shall show that k_d is continuous at $u(x)$. Assume it is not. For every $e \geq 0$, $u_{[d+e]}$ is more risk averse than $u_{[d]}$, so there is a strictly increasing concave function $k_{d,e} : u_{[d]}(I) \to I\!R$ such that $u_{[d+e]} = k_{d,e} \circ u_{[d]}$ on $u_{[d]}(I)$. Since

$$k_{[d+e]} \circ u = u_{[d+e]} = k_{d,e} \circ u_{[d]} = k_{d,e} \circ k_d \circ u$$

it follows that

$$k_{[d+e]} = k_{d,e} \circ k_d$$

Since, by assumption, k_d is not continuous at $u(x)$, it follows from (1) of Lemma 1 that $k_{[d+e]}$ is not continuous at $u(x)$. Since $k_{[d+e]} \circ u = u_{[d+e]}$, this implies that $u_{[d+e]}$ is not continuous at x, which means that u is not continouos at $x + d + e$. This is true of all $e \geq 0$, contradicting the fact that u is continuous almost everywhere.

Hence, k_d is continuous at $u(x)$. This is true of all $d \geq 0$, so so u is continuous $x + d$ for all $d \geq 0$. Since u is continuous almost everywhere, it follows that it is continuous everywhere.

Second, we show that u is differentiable with $u' > 0$. Let x be a point in I where u is differentiable.

Given $d \geq 0$, there is, as above, a strictly increasing concave function $k_d : u(I) \to I\!R$ such that $u_{[d]} = k_d \circ u$ on $u(I)$. Since u is continuous, $u(I)$ is an interval.

Since k_d is concave, if $u'(x) = 0$, then $u_{[d]}$ is differentiable at x with $u'_{[d]}(x) = 0$, and so u is differentiable at $x + d$ with $u'(x + d) = 0$. This would be true of all $d \geq 0$, contradicting the fact that u is strictly increasing. Hence, $u'(x) > 0$.

We shall show that k_d is differentiable at $u(x)$. Assume it is not. For every $e \geq 0$, there is, as above, a strictly increasing concave function $k_{d,e} : u_{[d]}(I) \to$

$\mathbb{R}$ such that $u_{[d+e]} = k_{d,e} \circ u_{[d]}$ on I. Since, as above,

$$k_{[d+e]} = k_{d,e} \circ k_d$$

it follows from (2) of Lemma 1 that $k_{[d+e]}$ is not differentiable at $u(x)$. Since $k_{[d+e]} \circ u = u_{[d+e]}$ and u is differentiable at x, this implies that $u_{[d+e]}$ is not differentiable at x. Hence, u is not differentiable at $x + d + e$. This is true of all $e \geq 0$, contradicting the fact that u is differentiable almost everywhere.

So, k_d is differentiable at $u(x)$. This is true of all $d \geq 0$, so u is differentiable at $x + d$ with $u'(x + d) > 0$, all such c. Since u is differentiable almost everywhere, it follows that it is differentiable everywhere.

The proof for the case of decreasing absolute risk aversion is identical, except that $d \geq 0$ is replaced by $d \leq 0$ and $e \geq 0$ is replaced by $e \leq 0$.

□

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