Parametric Characterizations of Risk Aversion and Prudence

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Abstract

Our first main result says whether one decision maker is more risk averse than another can be determined from their attitudes toward a given two-parameter family of risks. When all risks belong to this family, risk aversion can be compared even when initial wealth is random. Our second main result solves a long-standing problem in mean-variance analysis: what is the interpretation of the concavity of utility as a function of mean and variance? We show that in the case of normal distributions, this utility function is concave if and only if the agent has decreasing prudence.

1 Introduction

This paper investigates the concepts of risk aversion and prudence in situations where the risks in question are drawn from a two-parameter family of random variables parametrized by their mean and standard deviation.

A behavioral definition of comparative risk aversion says that one expected-utility maximizing decision maker is more risk averse than another if any risky prospect that he prefers to a sure outcome will also be preferred by the other. In order to determine whether he is indeed more risk averse than his colleague, it is, however, not necessary to check their behavior in all possible situations of choice between a risky prospect and a sure outcome. It is well known, for example, that it is sufficient to consider their behavior in situations involving binary lotteries.

Our first main result is a new criterion along these lines. It says that in order to determine whether one decision maker is more risk averse than another, it is, alternatively, sufficient to consider the decision makers' attitudes toward a given two-parameter family of gambles. Furthermore, if one of them is indeed more risk averse, then any two-parameter gamble that he prefers to a gamble with lower but not necessarily zero risk will also be preferred by his colleague. In other words, comparative risk aversion is extended to situations of "background risk" or random initial wealth, as in Kihlstrom, Roemer and Williams (1981), provided that the risks belong to the same two-parameter family.

Given the two-parameter family of random variables, expected utility becomes a function of mean and standard deviation. Because comparisons of risk aversion are determinate from considering only this family of risks, risk aversion can be measured in terms of standard deviation and mean, and indeed it can be measured by the marginal rate of substitution between mean and standard deviation, which is the slope of an indifference curve.

Our second main result is an economic interpretation, in the case of normal distributions, of the concavity of a utility function which is a function of variance and mean. It turns out that concavity of this function is equivalent to decreasing prudence.

The concept of prudence was introduced by Kimball (1990). It measures the strength of the precautionary saving motive, or the motive to save when confronted with uncertainty about future income.

In order to make our point about concavity and decreasing prudence, we need to develop the concepts of comparative and decreasing prudence in the context of a two-parameter family of distributions. We make use of the fact that Kimball's theory of prudence is isomorphic to the Arrow-Pratt theory of risk aversion as applied to minus the derivative of the utility function. This allows us to transform our results about risk aversion into results about prudence.

The plan of the paper is the following. In Section 2, we study the concept of risk aversion. The findings about risk aversion in Section 2 are reinterpreted in terms of prudence in Section 3. Section 4 shows the equivalence of decreasing prudence and concavity of the utility function for variance and mean.

2 Risk Aversion

This section gives a new characterization of comparative risk aversion based on two-parameter families of distributions.

Let Y be a random variable, and let u be a utility function, defined on the entire real line.

Assumption 1 *The random variable Y and the utility function u satisfy the following:*

- *Y has mean zero and variance one*
- *For every $\sigma \geq 0$ and for every μ, the random variable $u(\mu + \sigma Y)$ is integrable*
- *u is twice continuously differentiable with $u' > 0$ and $u'' < 0$, and u'' is non-decreasing*

- *For every* $\sigma \geq 0$ *and for every* μ, *the random variable* $Y^2u''(\mu+\sigma Y)$ *is integrable*

Define a derived utility function of standard deviation and mean (σ, μ) by

$$U(\sigma, \mu) = Eu(\mu + \sigma Y)$$

Let

$$S(\sigma, \mu) = -U_\sigma(\sigma, \mu)/U_\mu(\sigma, \mu)$$

denote the slope of the investor's indifference curve in (σ, μ)-space at (σ, μ).

Let u_1, u_2 be utility functions satisfying Assumption 1, and let U_1 and U_2 be the corresponding utility functions of standard deviation and mean. Say that U_1 is *more variance averse than* U_2 if for all $\sigma \geq 0$, $h \geq 0$ and all μ, k, $U_1(\sigma, \mu) \leq U_1(\sigma + h, \mu + k)$ implies $U_2(\sigma, \mu) \leq U_2(\sigma + h, \mu + k)$. It is easily seen that U_1 is more variance averse than U_2 if and only if $S_1(\sigma, \mu) \geq S_2(\sigma, \mu)$ for all (σ, μ) with $\sigma \geq 0$, where S_i denotes the slope of the indifference curve of U_i, $i = 1, 2$.

Theorem 1 *Under Assumption 1,* u_1 *is more risk averse than* u_2 *if and only if* U_1 *is more variance averse than* U_2.

Before proving Theorem 1, we need two lemmas.

Lemma 1 U *is strictly concave, and the derivatives* U_μ, U_σ *and* $U_{\sigma\sigma}$ *exist, are continuous, and have these properties :*

- $U_\mu(\sigma, \mu) = Eu'(\mu + \sigma Y) > 0$
- $U_\sigma(\sigma, \mu) = E[Yu'(\mu + \sigma Y)]$, $U_\sigma(0, \mu) = 0$ *for* $\sigma = 0$, *and* $U_\sigma(\sigma, \mu) < 0$ *for* $\sigma > 0$
- $U_{\sigma\sigma}(\sigma, \mu) = E[Y^2u''(\mu + \sigma Y)] < 0$ *and* $U_{\sigma\sigma}(0, \mu) = u''(\mu)$

PROOF:

Except for the statements about the signs of U_σ, these results follow from Nielsen (1993, Proposition 6).

Since u' is strictly decreasing,

$$0 < E\left(1_{\{Y\geq 0\}} Y u'(\mu+\sigma Y)\right) < E\left(1_{\{Y\geq 0\}} Y\right) u'(\mu)$$

and

$$0 > E\left(1_{\{Y<0\}} Y u'(\mu+\sigma Y)\right) > E\left(1_{\{Y<0\}} Y\right) u'(\mu)$$

Hence,

$$\begin{aligned} U_\sigma(\sigma,\mu) &= E[Yu'(\mu+\sigma Y)] \\ &= E\left(1_{\{Y\geq 0\}} Y u'(\mu+\sigma Y) + 1_{\{Y<0\}} Y u'(\mu+\sigma Y)\right) \\ &< E\left(1_{\{Y\geq 0\}} Y\right) u'(\mu) + E\left(1_{\{Y<0\}} Y\right) u'(\mu) \\ &= 0 \end{aligned}$$

□

Define a derived utility function of variance and mean (v,μ) by

$$W(v,\mu) = U(\sqrt{v},\mu)$$

Lemma 2 *W is continuously differentiable (also at $v=0$) with $W_v < 0$ and $W_\mu > 0$, and $2W_v(0,\mu) = u''(\mu)$.*

PROOF:

It follows from Lemma 1 W_μ exists and is continous everywhere,

$$W_\mu(v,\mu) = U_\mu(\sqrt{v},\mu) > 0$$

and $W_v(v,\mu)$ exists and is continuous for $v > 0$:

$$W_v(v,\mu) = \frac{1}{2\sqrt{v}} U_\sigma(\sqrt{v},\mu) < 0$$

Recall that $U_\sigma(0,\mu) = 0$ and $U_{\sigma\sigma}(0,\mu) = u''(\mu)$. Therefore,

$$W_v(0,\mu) = \frac{1}{2}\frac{U_\sigma(\sqrt{v},\mu) - U_\sigma(0,\mu)}{\sqrt{v}} \to \frac{1}{2}U_{\sigma\sigma}(0,\mu) = \frac{1}{2}u''(\mu)$$

as $v \to 0$. This shows that $W_v(0,\mu)$ exists and equals

$$W_v(0,\mu) = u''(\mu)/2 < 0$$

It remains to show that W_v is continuous at $v = 0$. Let (v_n, μ_n) be a sequence such that $(v_n, \mu_n) \to (0, \mu)$. Then

$$W_v(v_n,\mu_n) - W_v(v_n,\mu) = \frac{1}{2\sqrt{v_n}}[U_\sigma(\sqrt{v_n},\mu_n) - U_\sigma(\sqrt{v_n},\mu)] \to 0$$

Since we have already shown that $W_v(v_n,\mu) \to W_v(0,\mu)$ it follows that $W_v(v_n,\mu_n) \to W_v(0,\mu)$.

□

It follows from Lemma 2 that the slope of the indifference curve in (v,μ)-space at $v = 0$ is

$$-\frac{W_v(0,\mu)}{W_\mu(0,\mu)} = -\frac{1}{2}\frac{u''(\mu)}{u'(\mu)}$$

or half of the Arrow-Pratt coefficient of absolute risk aversion. This was shown by Chipman (1973) in the special case where Y follows a normal distribution.

Proof of Theorem 1

First, suppose that U_1 is more variance averse than U_2. Then $S_1(\sigma,\mu) \geq S_2(\sigma,\mu)$ for all (σ,μ) with $\sigma \geq 0$. Since

$$-\frac{W_{1v}(\sigma^2,\mu)}{W_{1\mu}(\sigma^2,\mu)} = \frac{1}{2\sigma}S_1(\sigma,\mu) \geq \frac{1}{2\sigma}S_2(\sigma,\mu) = -\frac{W_{2v}(\sigma^2,\mu)}{W_{2\mu}(\sigma^2,\mu)}$$

for all $\sigma > 0$, it follows that

$$\frac{W_{1v}(\sigma^2,\mu)}{W_{1\mu}(\sigma^2,\mu)} \leq \frac{W_{2v}(\sigma^2,\mu)}{W_{2\mu}(\sigma^2,\mu)}$$

for all $\sigma \geq 0$ (including $\sigma = 0$). For $\sigma = 0$ this implies that

$$-\frac{u_1''(\mu)}{u_1'(\mu)} = -\frac{2W_{1v}(0,\mu)}{W_{1\mu}(0,\mu)} \geq -\frac{2W_{2v}(0,\mu)}{W_{2\mu}(0,\mu)} = -\frac{u_2''(\mu)}{u_2'(\mu)}$$

which says that u_1 is more risk averse than u_2.

Conversely, suppose that u_1 is more risk averse than u_2. Inspired by the proof of Pratt (1964, Theorem 7), we proceed as follows. Given parameters μ_0 and s, define functions v_1 and v_2 by

$$v_i(\sigma) = \frac{1}{u_i'(\mu_0)} E u_i(\mu_0 + \sigma(s+Y)) = \frac{1}{u_i'(\mu_0)} U_i(\sigma, \mu_0 + \sigma s)$$

for $\sigma \geq 0$, $i = 1, 2$. Then

$$v_1'(\sigma) - v_2'(\sigma) = E\left[(s+Y)\left(\frac{u_1'(\mu_0 + \sigma(s+Y))}{u_1'(\mu_0)} - \frac{u_2'(\mu_0 + \sigma(s+Y))}{u_2'(\mu_0)}\right)\right]$$

It follows from Pratt (1964, Equation (20)) that

$$\frac{u_1'(\mu_0 + x)}{u_1'(\mu_0)} - \frac{u_2'(\mu_0 + x)}{u_2'(\mu_0)}$$

has the same sign as x, for all x. Setting $x = s + Y$ in the expectation above, it follows that the expectation is non-positive, and hence, that $v_1'(\sigma) \leq v_2'(\sigma)$. Given (σ, μ) with $\sigma \geq 0$, set $s = S_2(\sigma, \mu)$ and $\mu_0 = \mu - s\sigma$. Then

$$v_2'(\sigma) = \frac{1}{u_2'(\mu_0)} \left[U_{2\sigma}(\sigma, \mu_0 + \sigma s) + s U_{2\mu}(\sigma, \mu_0 + \sigma s)\right] = 0$$

and hence

$$0 \geq v_1'(\sigma) = \frac{1}{u_1'(\mu_0)} \left[U_{1\sigma}(\sigma, \mu_0 + \sigma s) + s U_{1\mu}(\sigma, \mu_0 + \sigma s)\right]$$

which implies that

$$S_1(\sigma, \mu) = -\frac{U_{1\sigma}(\sigma, \mu)}{U_{1\mu}(\sigma, \mu)} \geq s = S_2(\sigma, \mu)$$

□

3 Prudence and Concavity of W

This section develops the concept of prudence in the context of a two-parameter family of distributions. First, we observe how comparative prudence and decreasing prudence can be expressed in terms of the derived utility functions of standard deviation and mean. We then show that in the case of normal distributions, the utility function has decreasing prudence if and only if the derived utility function of variance and mean is concave.

As in Kimball (1990), consider a two-date model where an investor can borrow and lend at a riskfree interest rate of zero. His current wealth w is known, and his future income z is a random variable. He chooses his savings x so as to maximize an objective function of the form

$$v(w-x)+Eu(x+z)$$

The first-order condition for optimal choice says

$$v'(w-x)=Eu'(x+z)$$

If the future income is zero for sure, then the investor's savings x are determined by

$$v'(w-x)=Eu'(x)=u'(x)$$

The prospect of the non-zero random future income z will increase his savings relative to the situation where z is zero for sure if and only if

$$Eu'(x+z)\leq u'(x)$$

where x is still the optimal savings from the zero-income situation.

Now u_1 is *more prudent* than u_2 if for all non-random savings levels x and all random future incomes z such that both $u_1'(x+z)$ and $u_2'(x+z)$ have finite expectation, $Eu_1'(x+z)\leq u_1'(x)$ implies $Eu_2'(x+z)\leq u_2'(x)$.

It is immediately seen, as observed by Kimball (1990), that u_1 is more prudent than u_2 if and only if the "utility function" $-u_1'$ is more risk averse than the "utility function" $-u_2'$.

The derived utility function for standard deviation and mean (σ, μ) corresponding to the "utility function" $-u'$ is $-U_\mu$, where U is the derived utility function for standard deviation and mean corresponding to u.

To express prudence in terms of the derived utility function, we need $-u'$ to satisfy Assumption 1, which will be the case if u satisfies the following assumption:

Assumption 2 *The random variable Y and the utility function u satisfy the following:*

- *u is three times continuously differentiable with $u' > 0$, $u'' < 0$ and $u''' > 0$ and u''' is non-increasing*
- *The random variable $Y^2 u'''(\mu + \sigma Y)$ is integrable for every $\sigma \geq 0$ and every μ*

Remark 1 *Under Assumptions 1 and 2, u_1 is more prudent than u_2 if and only if $-U_{1\mu}$ is more variance averse than $-U_{2\mu}$.*

Proof: Follows from Theorem 1. □

Let $T(\sigma, \mu)$ denote the slope of the level curve of the function $-U_\mu$:

$$T(\sigma, \mu) = -U_{\mu\sigma}(\sigma, \mu)/U_{\mu\mu}(\sigma, \mu)$$

It turns out that these level curves coincide with iso-consumption curves. If u_1 and u_2 are two utility functions with derived utility functions U_1 and U_2 for standard deviation and mean, and if T_1 and T_2 denote the corresponding slopes of the level curves of the functions $-U_{1\mu}$ and $-U_{2\mu}$, then u_1 is more prudent than u_2 if and only if

$$T_1(\sigma, \mu) \geq T_2(\sigma, \mu)$$

for all (σ, μ) with $\sigma \geq 0$.

A utility function u has *decreasing prudence* if for each $k \geq 0$, u is more prudent than the utility function u^k defined by $u^k(\mu) = u(\mu + k)$. The derived utility function corresponding to u^k is $U^k(\sigma,\mu) = U(\sigma,\mu+k)$, and the slope of the indifference curve of $-U^k_\mu$ in (σ,μ)-space is $T^k(\sigma,\mu) = T(\sigma,\mu+k)$. Hence, it follows from Remark 1 that u has decreasing prudence if and only if U is more prudent than U^k for all $k \geq 0$, and if and only if $T(\sigma,\mu)$ is a decreasing function of μ.

Assumption 3 *The random variable Y and the utility function u satisfy the following:*

- *u is four times continuously differentiable with $u'''' \leq 0$*
- *Y is normally distributed*

In the case where Y follows a normal distribution, Chipman (1973) has shown that the second derivative of W with respect to v is

$$W_{vv} = \frac{1}{4} E u''''(\mu + \sqrt{v}Y).$$

Thus the assumption above ensures that $W_{vv} \leq 0$. Recall that Lemma 2 ensures that $W_{\mu\mu} < 0$.

Theorem 2 *Under Assumptions 1–3, the following conditions are equivalent:*

1. *W is concave*
2. *u exhibits decreasing prudence*
3. *The matrix*

$$Q = \begin{pmatrix} u'' & u''' \\ u''' & u'''' \end{pmatrix}$$

is negative semi-definite

PROOF:

(1) equivalent to (2) : When Y is normally distributed, the derived utility function U satisfies the differential equation $U_\sigma = \sigma U_{\mu\mu}$, (see Chipman (1973)), and the derivatives of U and W are related by $U_\sigma = 2\sigma W_v$ and $U_\mu = W_\mu$. So, $2W_v = W_{\mu\mu}$. We know that u exhibits decreasing prudence if and only if $T(\sigma, \mu)$ is a decreasing function of μ. Since

$$T(\sigma, \mu) = -\frac{U_{\mu\sigma}}{U_{\mu\mu}} = -2\sigma\frac{W_{\mu v}}{W_{\mu\mu}},$$

$T(\sigma, \mu)$ is decreasing in μ if and only if $-W_{\mu v}/W_{\mu\mu}$ is decreasing in μ. The derivative with respect to μ is

$$-\frac{W_{\mu v\mu}W_{\mu\mu} - W_{\mu v}W_{\mu\mu\mu}}{W_{\mu\mu}^2} = -\frac{2W_{vv}W_{\mu\mu} - W_{\mu v}2W_{v\mu}}{W_{\mu\mu}^2} = -\frac{W_{vv}W_{\mu\mu} - W_{\mu v}W_{v\mu}}{W_{\mu\mu}^2}.$$

This derivative is non-positive if and only if $W_{vv}W_{\mu\mu} - W_{\mu v}W_{v\mu} \geq 0$, which means that the determinant of the matrix

$$\begin{pmatrix} W_{vv} & W_{v\mu} \\ W_{\mu v} & W_{\mu\mu} \end{pmatrix}$$

is non-negative.

Since $W_{vv} \leq 0$ and $W_{\mu\mu} < 0$, the determinant is non-negative if and only if the matrix is negative semi-definite, which it is if and only if W is concave.

(2) equivalent to (3) : This follows directly by differentiating the coefficient of prudence, $-u'''/u''$.

□

Concavity of W is a stronger requirement than concavity of U. Allingham (1991), in his study of existence of equilibrium in CAPM, assumed that W is concave, whereas Nielsen (1990, 1992) assumed only that U is quasi-concave or, in some cases, that U is concave.

Chipman (1973, Theorem 3(c)) showed that condition (3) in Theorem 2 is sufficient for W to be concave, but he did not give an economic interpretation. The theorem shows that Chipman's condition is both necessary and sufficient, and our equivalent condition (2) provides the economic interpretation.

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